

The $d = 6$ trace anomaly from quantum field theory four-loop graphs in one dimension.

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Abstract

We calculate the integrated trace anomaly for a real spin-0 scalar field in six dimensions in a torsionless curved space without a boundary. We use a path integral approach for a corresponding supersymmetric quantum mechanical model. Weyl ordering the corresponding Hamiltonian in phase space, an extra two-loop counterterm $\frac{1}{8} \left(R + g^{ij} \Gamma_{ki}^l \Gamma_{lj}^k \right)$ is produced in the action. Applying a recursive method we evaluate the components of the metric tensor in Riemann normal coordinates in six dimensions and construct the interaction Lagrangian density by employing the background field method. The calculation of the anomaly is based on the end-point scalar propagator and not on the string inspired center-of-mass propagator which gives incorrect results for the local trace anomaly. The manipulation of the Feynman diagrams is partly relied on the factorization of four dimensional subdiagrams and partly on a brute force computer algebra program developed to serve this specific purpose. The computer program enables one to perform index contractions of twelve quantum fields (10395 in the present case) a task which cannot be accomplished otherwise. We observe that the contribution of the disconnected diagrams is no longer proportional to the two dimensional trace anomaly (which vanishes in four dimensions). The integrated trace anomaly is finally expressed in terms of the 17 linearly independent scalar monomials constructed out of covariant derivatives and Riemann tensors.

KEYWORDS: Weyl anomaly in six dimensions, Riemannian geometry, Computational techniques, Path integral methods

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1 Introduction

The trace anomaly, namely the breaking of classical conformal invariance of gravity actions under Weyl rescaling of the metric:

$$g_{ij}(x) \rightarrow \Omega(x)g_{ij}(x), \quad (1)$$

has a long history with numerous applications and implications to high energy physics, general relativity and statistical mechanics. The literature on this subject is vast and in this brief review we will only concentrate on those aspects which are associated with the present problem. For a historical review the reader is advised to consult [1].

Interesting different methods have been developed to investigate and calculate Weyl anomaly in four and higher dimensions. The authors in [2] were able to express the integrated trace anomaly in four dimensions as a linear combination of two invariants, the square of Weyl tensor:

$$C_{ijkl}C^{ijkl} = R_{ijkl}^2 - 2R_{ij}^2 + \frac{1}{3}R^2 \quad (2)$$

and the only parity even candidate:

$$\mathcal{E}_4 = {}^*R_{ijkl}{}^*R^{ijkl} = R_{ijkl}^2 - 4R_{ij}^2 + R^2 \quad (3)$$

which is proportional to the well-known Gauss-Bonnet topological density and $*$ denotes the dual. To be more concrete, the gravitational contribution to the anomaly depends on only two constants (call them α and β) and is expressed as

$$g^{ij} \langle T_{ij} \rangle = \alpha \left(C^{ijkl}C_{ijkl} + \frac{2}{3}\square R \right) + \beta \mathcal{E}_4. \quad (4)$$

The numerical values of the constants are: $\alpha = \frac{1}{\pi^2} \frac{1}{30} \frac{1}{64}$, $\beta = -\frac{1}{\pi^2} \frac{1}{90} \frac{1}{64}$ and can also be found using the Feynman diagram scheme in [17].

It was soon realised [3] that these invariants were manifested in the t -independent b_2 coefficient of the Schwinger-De Witt asymptotic expansion of the heat kernel of the appropriate differential operators. The four (and partially the six) dimensional anomaly was later rederived by Bonora et. al. [4] who established the connection between Weyl anomalies and cocycles by relying on a cohomological method and using the Wess-Zumino consistency condition. Although these authors explicitly specified the invariants (see Appendix A.2 for their classification) they did not express the six dimensional trace anomaly in terms of these invariants. From the representation theory point of view Fulling et. al. [5] were also able to determine the number of independent scalar monomials of each order and degree up to twelve in derivatives of the metric. The first explicit result was given in the literature by Gilkey [7] and later by Avramidi [8], who by making some modifications and innovations to the “heat kernel” or “proper time” method, presented a new covariant nonrecursive procedure and found the one-loop effective action in the presence of arbitrary background fields in six and eight curved space-time dimensions. Later Deser and Schwimmer [6] re-examined the different origin of the topological (type-A) versus local conformal scalar polynomials involving powers of the Weyl tensor and its derivatives (type-B) contributions to the anomaly in general dimensions.

In the present work we follow the supersymmetric quantum mechanical approach first pioneered by Alvarez-Gaumé and Witten [9] and used to compute chiral anomalies. According to this method the operators γ_5 , ∇_μ , x_μ , γ^μ are represented by operators of a corresponding

quantum mechanical model, and by turning these operator expressions into path integrals, one finds that anomalies of quantum field theories can be written in terms of Feynman diagrams for certain sigma models on the worldline. Bastianelli and van Nieuwenhuizen applied this method to trace anomalies [10]. These authors used mode regularization, a scheme widely used at the time. Subsequent work by de Boer et. al [11] showed how to use time-slicing and gave a completely and unambiguous derivation of trace anomalies in terms of path integrals, using as input Einstein Hamiltonians. In this article we apply the regularization method of [11] to the calculation of trace anomalies, following the set up of [10].

The outline of the paper is as follows. In the next section we present explicitly one of the basic ingredients of the background field method, namely the expansion of the metric tensor components in six dimensions in Riemann normal coordinates. The method we use is recursive and enables one with the application of (9) and (11) to determine the expansion of the metric tensor up to the desired order.

Section 3 begins with a very rapid introduction to trace anomalies from the one-dimensional path integral point of view. We write down the interaction Lagrangian density and the propagators of the fields involved.

Section 4 is devoted to the calculation of the perturbative expansion depicting at the same time the Feynman diagrams associated to each vertex.

A vital tool in our journey is a computer algebra algorithm which proved to be very efficient especially in finding the contribution of I_6 vertex with twelve quantum fields. Thus section 5 is devoted to a brief description of this program. We illustrate it's capabilities by applying it to the I_7 interaction with eight quantum fields.

Our conclusions are given in section 6. Several appendices follow to assist to a deeper understanding of the technical obstructions the reader might face.

2 The recursive expansion of the metric components in RNC in six space-time dimensions

Before embarking on the background field method we discuss the expansion of the metric tensor in Riemann normal coordinates (RNC).

RNC have the appealing feature that the geodesics passing through the origin have the same form as the equations of straight lines passing through the origin of a Cartesian system of coordinates in Euclidean geometry [12]. Locally no two geodesics through a point \mathcal{P} intersect at another point, and the power series solution of the geodesic equation is:

$$y^l = \xi^{i_1} s + \sum_{k=2}^{\infty} \frac{1}{k!} \left(\bar{\Gamma}_{i_1 i_2 \dots i_k}^l \right)_{\mathcal{P}} \xi^{i_1} \xi^{i_2} \dots \xi^{i_k} s^k. \quad (5)$$

where $\left(\bar{\Gamma}_{i_1 i_2 \dots i_k}^l \right)_{\mathcal{P}}$ are the “*generalized Christoffel symbols*” at the point \mathcal{P} and the geodesics through \mathcal{P} which are straight lines are defined in terms of the arc length s by:

$$y^l = \xi^l s. \quad (6)$$

By induction, one can easily prove that:

$$\partial_{(i_1} \partial_{i_2} \dots \partial_{i_{k-2}} \bar{\Gamma}_{i_{k-1} i_k)}^l = 0 \quad (7)$$

Paraphrasing eq. (7), one can state that all symmetric derivatives of the affine connection vanish at the origin in RNC.

In general a covariant second rank tensor field on a manifold can be expanded according to:

$$T_{k_1 k_2}(\tilde{\phi}) = T_{k_1 k_2}(\phi) + \sum_{n=1}^{\infty} \frac{1}{n!} \left[\frac{\partial}{\partial \xi_{i_1}} \frac{\partial}{\partial \xi_{i_2}} \cdots \frac{\partial}{\partial \xi_{i_n}} \right] T_{k_1 k_2}(\phi) \xi_{i_1} \xi_{i_2} \cdots \xi_{i_n}. \quad (8)$$

The coefficients of the Taylor expansion are tensors and can be expressed in terms of the components R_{mnp}^l of the Riemann curvature tensor and the covariant derivatives $D_k T_{lm}$ and $D_k R_{mnp}^l$. Without much effort one can prove that:

$$\begin{aligned} \partial_{(i_1} \partial_{i_2} \cdots \partial_{i_{n-1}} \bar{\Gamma}_{i_n)k}^l = & - \left(\frac{n-1}{n+1} \right) \left[D_{(i_1} D_{i_2} \cdots D_{i_{n-2}} \bar{R}_{i_{n-1} k i_n)}^l \right. \\ & + \partial_{(i_1} \partial_{i_2} \cdots \partial_{i_{n-2}} \left(\bar{\Gamma}_{i_{n-1} k}^\alpha \bar{\Gamma}_{i_n) \alpha}^l \right) \\ & - \partial_{(i_1} \partial_{i_2} \cdots \partial_{i_{n-3}} \left(\bar{\Gamma}_{i_{n-2} \alpha}^l \bar{R}_{i_{n-1} k i_n)}^\alpha - l \leftrightarrow \alpha, \alpha \leftrightarrow k \right) \\ & - \partial_{(i_1} \partial_{i_2} \cdots \partial_{i_{n-4}} \left(\bar{\Gamma}_{i_{n-3} \alpha}^l D_{i_{n-2}} \bar{R}_{i_{n-1} k i_n)}^\alpha - l \leftrightarrow \alpha, \alpha \leftrightarrow k \right) \\ & \vdots \\ & \left. - \left(\partial_{(i_1} \bar{\Gamma}_{i_2 \alpha}^l D_{i_3} \cdots D_{i_{n-2}} \bar{R}_{i_{n-1} k i_n)}^\alpha - l \leftrightarrow \alpha, \alpha \leftrightarrow k \right) \right] \end{aligned} \quad (9)$$

where the interchange of covariant and contravariant indices act independently and symmetrization acts only on i indices. Expression (9) reproduces for various values of n ¹ the following results:

$$\begin{aligned} \partial_{(i_1} \bar{\Gamma}_{i_2)k}^l = & - \frac{1}{3} \bar{R}_{(i_1 k i_2)}^l \\ \partial_{(i_1} \partial_{i_2} \bar{\Gamma}_{i_3)k}^l = & - \frac{1}{2} D_{(i_1} \bar{R}_{i_2 k i_3)}^l \\ \partial_{(i_1} \partial_{i_2} \partial_{i_3} \bar{\Gamma}_{i_4)k}^l = & - \frac{3}{5} \left[D_{(i_1} D_{i_2} \bar{R}_{i_3 k i_4)}^l + \frac{2}{9} \bar{R}_{(i_1 i_2 \alpha}^l \bar{R}_{i_3 i_4) k}^\alpha \right] \\ \partial_{(i_1} \partial_{i_2} \partial_{i_3} \partial_{i_4} \bar{\Gamma}_{i_5)k}^l = & - \frac{2}{3} \left[D_{(i_1} D_{i_2} D_{i_3} \bar{R}_{i_4 k i_5)}^l - D_{(i_1} \bar{R}_{i_2 k i_3}^\alpha \bar{R}_{i_4 i_5) \alpha}^l \right] \\ \partial_{(i_1} \partial_{i_2} \partial_{i_3} \partial_{i_4} \partial_{i_5} \bar{\Gamma}_{i_6)k}^l = & - \frac{5}{7} \left[D_{(i_1} \cdots D_{i_4} \bar{R}_{i_5 k i_6)}^l \right. \\ & - \frac{1}{5} \left(7 D_{(i_1} D_{i_2} \bar{R}_{i_3 \alpha i_4}^l \bar{R}_{i_5 i_6) k}^\alpha + D_{(i_1} D_{i_2} \bar{R}_{i_3 k i_4}^\alpha \bar{R}_{i_5 i_6) \alpha}^l \right) \\ & \left. + \frac{3}{2} D_{(i_1} \bar{R}_{i_2 k i_3}^\alpha D_{i_4} \bar{R}_{i_5 \alpha i_6)}^l - \frac{16}{45} \bar{R}_{(i_1 i_2 \alpha}^l \bar{R}_{i_3 i_4 \beta}^\alpha \bar{R}_{i_5 i_6) k}^\beta \right] \\ & \dots \end{aligned} \quad (10)$$

The coefficients of(8) can be rewritten as:

$$\begin{aligned} \partial_{(i_1} \partial_{i_2} \cdots \partial_{i_n}) \bar{T}_{k_1 k_2} = & D_{(i_1} D_{i_2} \cdots D_{i_n}) \bar{T}_{k_1 k_2} \\ & + \partial_{(i_1} \partial_{i_2} \cdots \partial_{i_{n-1}} \left[\bar{\Gamma}_{i_n k_1}^\alpha \bar{T}_{\alpha k_2} + k_1 \leftrightarrow k_2 \right] \\ & + \partial_{(i_1} \partial_{i_2} \cdots \partial_{i_{n-2}} \left[\bar{\Gamma}_{i_{n-1} k_1}^\alpha D_{i_n}) \bar{T}_{\alpha k_2} + k_1 \leftrightarrow k_2 \right] \end{aligned}$$

¹In [13] there is a misprint for the $n = 4$ case. A minus sign is needed in front of the $\frac{2}{9}$ -term.

$$\begin{aligned}
& \vdots \\
& + \partial_{(i_1} \partial_{i_2} \left[\bar{\Gamma}_{i_3 k_1}^\alpha D_{i_4} \cdots D_{i_n} \bar{T}_{\alpha k_2} + k_1 \leftrightarrow k_2 \right] \\
& + \partial_{(i_1} \left[\bar{\Gamma}_{i_2 k_1}^\alpha D_{i_3} \cdots D_{i_n} \bar{T}_{\alpha k_2} + k_1 \leftrightarrow k_2 \right]. \tag{11}
\end{aligned}$$

Expressions (9) and (11) compose the building blocks of the current recursive method which produces the following results for different values of n:

$$\begin{aligned}
\partial_{(i_1} \partial_{i_2)} \bar{T}_{k_1 k_2} &= D_{(i_1} D_{i_2)} \bar{T}_{k_1 k_2} - \frac{1}{3} \left[\bar{R}_{(i_1 k_1 i_2)}^\rho \bar{T}_{\rho k_2} + k_1 \leftrightarrow k_2 \right] \\
\partial_{(i_1} \partial_{i_2} \partial_{i_3)} \bar{T}_{k_1 k_2} &= D_{(i_1} D_{i_2} D_{i_3)} \bar{T}_{k_1 k_2} \\
&+ \left[\partial_{(i_1} \partial_{i_2} \bar{\Gamma}_{i_3 k_1)}^\rho \bar{T}_{\rho k_2} + 2 \partial_{(i_1} \bar{\Gamma}_{i_2 k_1)}^\rho D_{i_3)} \bar{T}_{\rho k_2} + k_1 \leftrightarrow k_2 \right] \\
&- \frac{1}{3} \left(\bar{R}_{(i_1 k_1 i_2)}^\rho D_{i_3)} \bar{T}_{\rho k_2} + k_1 \leftrightarrow k_2 \right) \\
\partial_{(i_1} \partial_{i_2} \partial_{i_3} \partial_{i_4)} \bar{T}_{k_1 k_2} &= D_{(i_1} D_{i_2} D_{i_3} D_{i_4)} \bar{T}_{k_1 k_2} \\
&+ \left[\partial_{(i_1} \partial_{i_2} \partial_{i_3} \bar{\Gamma}_{i_4 k_1)}^\rho \bar{T}_{\rho k_2} + 3 \partial_{(i_1} \partial_{i_2} \bar{\Gamma}_{i_3 k_1)}^\rho D_{i_4)} \bar{T}_{\rho k_2} \right. \\
&+ \left. 3 \partial_{(i_1} \bar{\Gamma}_{i_2 k_1)}^\rho \left(D_{i_3} D_{i_4)} \bar{T}_{\rho k_2} - \frac{1}{3} \left(\bar{R}_{i_3 \rho i_4)}^\sigma \bar{T}_{\sigma k_2} + \rho \leftrightarrow k_2 \right) \right) + k_1 \leftrightarrow k_2 \right] \\
\partial_{(i_1} \partial_{i_2} \partial_{i_3} \partial_{i_4} \partial_{i_5)} \bar{T}_{k_1 k_2} &= D_{(i_1} D_{i_2} D_{i_3} D_{i_4} D_{i_5)} \bar{T}_{k_1 k_2} \\
&- \left[\frac{10}{3} \bar{R}_{(i_1 k_1 i_2)}^\alpha D_{i_3} \cdots D_{i_5)} \bar{T}_{\alpha k_2} + 5 D_{(i_1} \bar{R}_{i_2 k_1 i_3)}^\alpha D_{i_4} D_{i_5)} \bar{T}_{\alpha k_2} \right. \\
&+ 3 D_{(i_1} D_{i_2} \bar{R}_{i_3 k_1 i_4)}^\alpha D_{i_5)} \bar{T}_{\alpha k_2} + \frac{2}{3} D_{(i_1} D_{i_2} D_{i_3} \bar{R}_{i_4 k_1 i_5)}^\alpha \bar{T}_{\alpha k_2} \\
&- \frac{2}{3} D_{(i_1} \bar{R}_{i_2 k_1 i_3)}^\rho \bar{R}_{i_4 i_5) \rho}^\alpha \bar{T}_{\alpha k_2} + D_{(i_1} \bar{R}_{i_2 k_1 i_3)}^\alpha \left(\bar{R}_{i_4 \alpha i_5)}^\rho \bar{T}_{\rho k_2} + \alpha \leftrightarrow k_2 \right) \\
&+ \frac{2}{3} \left(D_{(i_1} \bar{R}_{i_2 \alpha i_3)}^\rho \bar{T}_{\rho k_2} + \alpha \leftrightarrow k_2 \right) \bar{R}_{i_4 i_5) k_1}^\alpha \pm k_1 \leftrightarrow k_2 \left. \right] \\
\partial_{(i_1} \partial_{i_2} \partial_{i_3} \partial_{i_4} \partial_{i_5} \partial_{i_6)} \bar{T}_{k_1 k_2} &= D_{(i_1} \cdots D_{i_6)} \bar{T}_{k_1 k_2} \\
&+ \left[\partial_{(i_1} \cdots \partial_{i_5} \bar{\Gamma}_{i_6 k_1)}^\alpha \bar{T}_{\alpha k_2} + 6 \partial_{(i_1} \cdots \partial_{i_4} \bar{\Gamma}_{i_5 k_1)}^\alpha D_{i_6)} \bar{T}_{\alpha k_2} \right. \\
&+ 15 \partial_{(i_1} \cdots \partial_{i_3} \bar{\Gamma}_{i_4 k_1)}^\alpha D_{i_5} D_{i_6)} \bar{T}_{\alpha k_2} + 20 \partial_{(i_1} \partial_{i_2} \bar{\Gamma}_{i_3 k_1)}^\alpha D_{i_4} \cdots D_{i_6)} \bar{T}_{\alpha k_2} \\
&+ 14 \partial_{(i_1} \bar{\Gamma}_{i_2 k_1)}^\alpha D_{i_3} \cdots D_{i_6)} \bar{T}_{\alpha k_2} \\
&+ 10 \partial_{(i_1} \cdots \partial_{i_3} \bar{\Gamma}_{i_4 k_1)}^\alpha \left(\partial_{i_5} \bar{\Gamma}_{i_6) \alpha}^\rho \bar{T}_{\rho k_2} + \alpha \leftrightarrow k_2 \right) \\
&+ 10 \partial_{(i_1} \partial_{i_2} \bar{\Gamma}_{i_3 k_1)}^\alpha \left(\partial_{i_4} \partial_{i_5} \bar{\Gamma}_{i_6) \alpha}^\rho \bar{T}_{\rho k_2} + \alpha \leftrightarrow k_2 \right) \\
&+ 36 \partial_{(i_1} \partial_{i_2} \bar{\Gamma}_{i_3 k_1)}^\alpha \left(\partial_{i_4} \bar{\Gamma}_{i_5 \alpha}^\rho D_{i_6)} \bar{T}_{\rho k_2} + \alpha \leftrightarrow k_2 \right) \\
&+ 5 \partial_{(i_1} \bar{\Gamma}_{i_2 k_1)}^\alpha \left(\partial_{i_3} \cdots \partial_{i_5} \bar{\Gamma}_{i_6) \alpha}^\rho \bar{T}_{\rho k_2} + \alpha \leftrightarrow k_2 \right) \\
&+ 24 \partial_{(i_1} \bar{\Gamma}_{i_2 k_1)}^\alpha \left(\partial_{i_3} \partial_{i_4} \bar{\Gamma}_{i_5 \alpha}^\rho D_{i_6)} \bar{T}_{\rho k_2} + \alpha \leftrightarrow k_2 \right) \\
&+ 45 \partial_{(i_1} \bar{\Gamma}_{i_2 k_1)}^\alpha \left(\partial_{i_3} \bar{\Gamma}_{i_4 \alpha}^\rho D_{i_5} D_{i_6)} \bar{T}_{\rho k_2} + \alpha \leftrightarrow k_2 \right) \\
&+ 15 \left[\partial_{(i_1} \bar{\Gamma}_{i_2 k_1)}^\alpha \partial_{i_3} \bar{\Gamma}_{i_4 \alpha}^\rho \left(\partial_{i_5} \bar{\Gamma}_{i_6) \rho}^\sigma \bar{T}_{\sigma k_2} + \rho \leftrightarrow k_2 \right) + \alpha \leftrightarrow k_2 \right] \\
&+ \left. \partial_{(i_1} \bar{\Gamma}_{i_2 k_1)}^\alpha D_{i_3} \cdots D_{i_6)} \bar{T}_{\alpha k_2} + k_1 \leftrightarrow k_2 \right]. \tag{12}
\end{aligned}$$

If the second rank tensor with components $\bar{T}_{k_1 k_2}$ is replaced by the metric components $\bar{g}_{k_1 k_2}$ then the related covariant derivatives (provided we deal with a torsion free affine connection)

vanish and the above expressions are simplified. One could derive for $n = 5$ the result:

$$\partial_{(i_1} \cdots \partial_{i_5)} \bar{g}_{k_1 k_2} = \frac{4}{3} \left[D_{i_1} \cdots D_{i_3} \bar{R}_{k_1 i_4 i_5 k_2} + 2 \left(D_{i_1} \bar{R}_{k_1 i_2 i_3 \rho} \bar{R}_{i_4 i_5 k_2}^\rho + k_1 \leftrightarrow k_2 \right) \right]. \quad (13)$$

On the other hand for $n = 6$ one gets:

$$\begin{aligned} \partial_{(i_1} \partial_{i_2} \partial_{i_3} \partial_{i_4} \partial_{i_5} \partial_{i_6)} \bar{g}_{k_1 k_2} = & \frac{10}{7} D_{(i_1} \cdots D_{i_4} \bar{R}_{k_1 i_5 i_6) k_2} \\ & + \frac{34}{7} \left(D_{(i_1} D_{i_2} \bar{R}_{k_1 i_3 i_4 \rho} \bar{R}_{i_5 i_6) k_2}^\rho + k_1 \leftrightarrow k_2 \right) \\ & + \frac{55}{7} D_{(i_1} \bar{R}_{k_1 i_2 i_3 \rho} D_{i_4} \bar{R}_{i_5 i_6) k_2}^\rho + \frac{16}{7} \bar{R}_{k_1 (i_1 i_2 \rho} \bar{R}_{i_3 i_4 l}^\rho \bar{R}_{i_5 i_6) k_2}^l. \end{aligned} \quad (14)$$

Thus, plugging into(8) expressions(13) and(14) we end up with the following expansion of the metric tensor in RNC:

$$\begin{aligned} g_{k_1 k_2} = & \bar{g}_{k_1 k_2} + \frac{1}{2!} \frac{2}{3} \bar{R}_{k_1 i_1 i_2 k_2} \xi^{i_1} \xi^{i_2} \\ & + \frac{1}{3!} D_{i_1} \bar{R}_{k_1 i_2 i_3 k_2} \xi^{i_1} \cdots \xi^{i_3} \\ & + \frac{1}{4!} \frac{6}{5} \left[D_{i_1} D_{i_2} \bar{R}_{k_1 i_3 i_4 k_2} + \frac{8}{9} \bar{R}_{k_1 i_1 i_2 m} \bar{R}_{i_3 i_4 k_2}^m \right] \xi^{i_1} \cdots \xi^{i_4} \\ & + \frac{1}{5!} \frac{4}{3} \left[D_{i_1} \cdots D_{i_3} \bar{R}_{k_1 i_4 i_5 k_2} + 2 \left(D_{i_1} \bar{R}_{k_1 i_2 i_3 \rho} \bar{R}_{i_4 i_5 k_2}^\rho + k_1 \leftrightarrow k_2 \right) \right] \xi^{i_1} \cdots \xi^{i_5} \\ & + \frac{1}{6!} \frac{10}{7} \left[D_{i_1} \cdots D_{i_4} \bar{R}_{k_1 i_5 i_6 k_2} + \frac{17}{5} \left(D_{i_1} D_{i_2} \bar{R}_{k_1 i_3 i_4 \rho} \bar{R}_{i_5 i_6 k_2}^\rho + k_1 \leftrightarrow k_2 \right) \right] \xi^{i_1} \cdots \xi^{i_6} \\ & + \frac{11}{2} D_{i_1} \bar{R}_{k_1 i_2 i_3 \rho} D_{i_4} \bar{R}_{i_5 i_6 k_2}^\rho + \frac{8}{5} \bar{R}_{k_1 i_1 i_2 \rho} \bar{R}_{i_3 i_4 l}^\rho \bar{R}_{i_5 i_6 k_2}^l \xi^{i_1} \cdots \xi^{i_6} \\ & + O(\xi^{i_1} \cdots \xi^{i_7}). \end{aligned} \quad (15)$$

The expression(15) will play a crucial role in calculating the contribution stemming from the quadratic Christoffel symbol term in the interaction Lagrangian. It is also in perfect agreement with the closed formula for these coefficients which are encoded in the integral representation of [14].

3 The integrated trace anomaly

Anomalies in (even) n -dimensional quantum field theories² are expressed in the Fujikawa [15] approach as³:

$$An_W = \lim_{\beta \rightarrow 0} Tr \left(\mathcal{J} e^{-\frac{\beta}{\hbar} \hat{\mathcal{R}}} \right) \quad (16)$$

where \mathcal{J} is the Jacobian $\frac{\partial \delta \tilde{\phi}(x)}{\partial \tilde{\phi}(y)} = f(x) \tilde{\phi}$ of the fields $\tilde{\phi}(y)$ and the regulator $\hat{\mathcal{R}}$ for consistent anomalies is uniquely determined [16]. For local Weyl anomalies and for real scalar fields one

² For a manifold having odd dimensionality one cannot form a scalar out of an odd number of derivatives.

³ We perform the usual redefinition of the scalar fields $\tilde{\phi} = g^{\frac{1}{4}} \phi$ which leaves the Jacobian invariant under the similarity transformation $g^{\frac{1}{4}}$.

finds that:

$$\begin{aligned}\hat{\mathcal{R}} &= \hat{H} - \frac{1}{2}\hbar^2\xi R(\hat{x}) \\ &= \frac{1}{2}g^{-\frac{1}{4}}(\hat{x})\hat{p}_ig^{ij}(\hat{x})g^{\frac{1}{2}}(\hat{x})\hat{p}_jg^{-\frac{1}{4}}(\hat{x}) - \frac{1}{2}\hbar^2\xi R(\hat{x}); \quad g(\hat{x}) = \det g_{ij}(\hat{x}).\end{aligned}\quad (17)$$

The second term in (17) is the well-known improvement potential term and the dimensionless coefficient $\xi = \frac{(n-2)}{n(n-1)}$ takes the value $\xi(n=6) = \frac{2}{15}$ in the present case. The Hamiltonian is Einstein invariant which means that \hat{H} commutes with the generator $\hat{G}(\xi) = \frac{1}{2i\hbar}(\hat{p}_i\xi^i(\hat{x}) + \xi^i(\hat{x})\hat{p}_i)$ of the infinitesimal target space diffeomorphisms $\hat{x}_i \rightarrow \hat{x}_i + \xi_i(\hat{x})$.

Consider a spin-0 field which lives on an n-dimensional compact Riemannian manifold equipped with its standard (metric-compatible, torsion-free) connection and having no boundary. In addition decompose the paths $x^i(\sigma)$ into a constant part x_0^i satisfying the free field equations and a quantum fluctuating $q^i(\sigma)$ one vanishing at the time boundaries. The Weyl anomaly for a real spin-0 field may be represented by the Euclidean quantum mechanical path integral [17]:

$$\begin{aligned}An_W(s=0, n) &= \lim_{\beta \rightarrow 0} Tr \left(f(x) e^{-\beta \hat{H}} \right) \\ &= \lim_{\beta \rightarrow 0} \left(\frac{1}{(2\pi\beta\hbar)^{n/2}} \int dx_0^i \prod_{i=1}^n \sqrt{g(x_0^i)} f(x_0^i) < e^{-\frac{1}{\hbar} S^{int}} > \right)\end{aligned}\quad (18)$$

where $f(x)$ is an arbitrary function, $-\frac{1}{\hbar}S^{int} = S_1^{int} + S_2^{int}$ and ⁴:

$$\begin{aligned}S_1^{in} &= \frac{1}{2\beta\hbar} \int_{-1}^0 [g_{ij}(x_0 + q) - g_{ij}(x_0)] (\dot{q}^i \dot{q}^j + b^i c^j + a^i a^j) d\sigma \\ &= \frac{1}{\beta\hbar} \int_{-1}^0 \left[\frac{1}{6} \bar{R}_{ii_1 i_2 j} q^{i_1} q^{i_2} + \frac{1}{12} D_{i_1} \bar{R}_{ii_2 i_3 j} q^{i_1} q^{i_2} q^{i_3} \right. \\ &\quad + \frac{3}{5!} \left(D_{i_1} D_{i_2} \bar{R}_{ii_3 i_4 j} + \frac{8}{9} \bar{R}_{ii_1 i_2 k} \bar{R}^k_{i_3 i_4 j} \right) q^{i_1} \dots q^{i_4} \\ &\quad + \frac{4}{6!} \left(D_{i_1} \dots D_{i_3} \bar{R}_{ii_4 i_5 j} + 2(D_{i_1} \bar{R}_{ii_2 i_3 k} \bar{R}^k_{i_4 i_5 j} + i \leftrightarrow j) \right) q^{i_1} \dots q^{i_5} \\ &\quad + \frac{5}{7!} \left(D_{i_1} \dots D_{i_4} \bar{R}_{ii_5 i_6 j} + \frac{17}{5} (D_{i_1} D_{i_2} \bar{R}_{ii_3 i_4 k} \bar{R}^k_{i_5 i_6 j} + i \leftrightarrow j) + \frac{11}{2} D_{i_1} \bar{R}_{ii_2 i_3 k} D_{i_4} \bar{R}^k_{i_5 i_6 j} \right. \\ &\quad \left. + \frac{8}{5} \bar{R}_{ii_1 i_2 k} \bar{R}^k_{i_3 i_4 l} \bar{R}^l_{i_5 i_6 j} \right) q^{i_1} \dots q^{i_6} + O(q^{i_1} \dots q^{i_7}) \left. \right] (\dot{q}^i \dot{q}^j + b^i c^j + a^i a^j) d\sigma\end{aligned}\quad (19)$$

$$\begin{aligned}S_2^{int} &= -\beta\hbar \frac{1}{8} \int_{-1}^0 \left[(1 - 4\xi) R + g^{ij} \Gamma^l_{ki} \Gamma^k_{lj} \right] (x_0 + q) d\tau \\ &= -\beta\hbar \frac{1}{8} \int_{-1}^0 \left[(1 - 4\xi) \left(\bar{R} + q^{i_1} D_{i_1} \bar{R} + \frac{1}{2!} q^{i_1} q^{i_2} D_{i_1} D_{i_2} \bar{R} + \frac{1}{3!} q^{i_1} \dots q^{i_3} D_{i_1} \dots D_{i_3} \bar{R} \right. \right. \\ &\quad \left. + \frac{1}{4!} q^{i_1} \dots q^{i_4} D_{i_1} \dots D_{i_4} \bar{R} \right) + \bar{g}^{ij} \partial_{i_1} \bar{\Gamma}^l_{ik} \partial_{i_2} \bar{\Gamma}^k_{lj} q^{i_1} q^{i_2} \\ &\quad \left. + \frac{1}{2!} \bar{g}^{ij} \left(\partial_{i_1} \partial_{i_2} \bar{\Gamma}^l_{ik} \partial_{i_3} \bar{\Gamma}^k_{lj} + \partial_{i_1} \bar{\Gamma}^l_{ik} \partial_{i_2} \partial_{i_3} \bar{\Gamma}^k_{lj} \right) q^{i_1} \dots q^{i_3} \right]\end{aligned}$$

⁴The expectation value $< \dots >$ means that all quantum fields must be contracted using the appropriate propagators. A bar over the various geometrical quantities indicates that they depend exclusively on x_0^i .

$$\begin{aligned}
& - \left(\frac{1}{3} \bar{R}^i{}_{i_1 i_2}{}^j \partial_{i_3} \bar{\Gamma}^l{}_{ik} \partial_{i_4} \bar{\Gamma}^k{}_{lj} - \frac{1}{4} \bar{g}^{ij} \partial_{i_1} \partial_{i_2} \bar{\Gamma}^l{}_{ik} \partial_{i_3} \partial_{i_4} \bar{\Gamma}^k{}_{lj} \right. \\
& - \left. \frac{\bar{g}^{ij}}{3!} \left(\partial_{i_1} \cdots \partial_{i_3} \bar{\Gamma}^l{}_{ik} \partial_{i_4} \bar{\Gamma}^k{}_{lj} + \partial_{i_1} \bar{\Gamma}^l{}_{ik} \partial_{i_2} \cdots \partial_{i_4} \bar{\Gamma}^k{}_{lj} \right) \right) q^{i_1} \cdots q^{i_4} \\
& + O(q^{i_1} \cdots q^{i_5}) \Big] d\tau.
\end{aligned} \tag{20}$$

As emphasized in [11], the above expression is the continuum limit of a rigorous discrete result.

In the action (19) (b^i, c^j) and α^i is a set of anticommuting and commuting Lee-Yang ghosts respectively [10]. Their existence is imposed by the integration over the momenta $p^i(\tau)$ in the transition from phase space to configuration space. A measure factor \sqrt{g} is then produced at each point of the path and by exponentiating it, introducing the Lee-Yang ghosts, we are led to a perfectly regular term in the action. The existence of ghost fields also removes ultraviolet divergencies at higher loops and as a consequence all integrals are finite.

The non-covariant \hbar^2 terms, R and $\Gamma\Gamma$, which are essential for the general coordinate invariance of the transition element are created [11] by Weyl ordering the Hamiltonian \hat{H} of (17). The contribution of these terms to the trace anomaly is found by first Taylor expanding them in RNC and then substituting the partial derivatives of the Christoffel symbols by the polynomials of R and DR :

$$\partial_{i_1} \bar{\Gamma}^l{}_{(i_2 k)} = -\frac{2}{3} \bar{R}^l{}_{(i_2 k) i_1} \tag{21}$$

$$\partial_{(i_1} \partial_{i_2} \bar{\Gamma}^l{}_{i_3) k} = -\frac{1}{2} D_{(i_1} \bar{R}^l{}_{i_2 k i_3)} \tag{22}$$

$$\partial_{(i_1} \partial_{i_2} \partial_{i_3} \bar{\Gamma}^l{}_{i_4) k} = -\frac{3}{5} \left[D_{(i_1} D_{i_2} \bar{R}^l{}_{i_3 k i_4)} + \frac{2}{9} \bar{R}^l{}_{(i_1 i_2 \alpha} \bar{R}^\alpha{}_{i_3 i_4) k} \right]. \tag{23}$$

The symmetrization of the various indices in the above identities is understood with the inclusion of a $\frac{1}{n!}$ factor. All other extra terms in (19) and (20) are produced by expanding the metric in Riemann normal coordinates with the help of (15).

Propagators are derived in closed form through the discretised configuration space path integrals via a midpoint rule. In this way ambiguities arising from products of distributions are resolved. Taking the continuum limits of the propagators one can read off the Feynman rules. One then finds that $\delta(\sigma - \tau)$ is to be considered as a Kronecker delta δ_{ij} and the propagators depend on the discrete Heaviside function $\theta_{ii} = \frac{1}{2}$. The continuous two point Green function may also be determined by the equation:

$$\frac{\partial^2}{\partial \sigma^2} \Delta(\sigma - \tau) = \delta(\sigma - \tau) \tag{24}$$

subjected to the boundary conditions on the interval $[-1, 0]$ (end-point approach):

$$\Delta(0, \tau) = \Delta(-1, \tau) = \Delta(\sigma, 0) = \Delta(\sigma, -1) = 0. \tag{25}$$

The Feynman propagator is then formally found to be:

$$\Delta(\sigma, \tau) = \sigma(\tau + 1) + (\tau - \sigma)\theta(\tau - \sigma). \tag{26}$$

The propagators of the various fields are proportional to $\beta\hbar$ and given by:

$$q^i(\sigma) q^j(\tau) = -\beta\hbar \bar{g}^{ij} \Delta(\sigma, \tau) \tag{27}$$

$$q^i(\sigma)\dot{q}^j(\tau) = -\beta\hbar\bar{g}^{ij}[\sigma + \theta(\tau - \sigma)] = -\beta\hbar\bar{g}^{ij}\Delta^\bullet(\sigma, \tau) \quad (28)$$

$$\dot{q}^i(\sigma)q^j(\tau) = -\beta\hbar\bar{g}^{ij}[\tau + \theta(\sigma - \tau)] = -\beta\hbar\bar{g}^{ij}\bullet\Delta(\sigma, \tau) \quad (29)$$

$$\dot{q}^i(\sigma)\dot{q}^j(\tau) = -\beta\hbar\bar{g}^{ij}[1 - \delta(\sigma - \tau)] = -\beta\hbar\bar{g}^{ij}\bullet\Delta^\bullet(\sigma, \tau) \quad (30)$$

$$b^i(\sigma)c^j(\tau) = -2\beta\hbar\bar{g}^{ij}\delta(\sigma - \tau) \quad (31)$$

$$a^i(\sigma)a^j(\tau) = +\beta\hbar\bar{g}^{ij}\delta(\sigma - \tau). \quad (32)$$

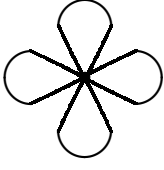
All other Wick contractions of the fields vanish. Regarding the vertices they may be read directly from the continuum S_{int} given by (19) and (20).

4 The Feynman diagrams and the associated contributions

Perturbative expansion of $\langle e^{-\frac{1}{\hbar}S_{int}} \rangle$, keeping only terms that cancel the $(\beta\hbar)^{-3}$ factor in the measure of the trace anomaly, provides us with the following distinct interactions as well as connected and disconnected diagrams having at most four loops ⁵:

1.

$$\begin{aligned} I_1 &= \frac{1}{\beta\hbar} \frac{5}{7!} \langle \int_{-1}^0 \left[D_{i_1} \cdots D_{i_4} \bar{R}_{ii_5i_6j} + \frac{17}{5} (D_{i_1} D_{i_2} \bar{R}_{ii_3i_4k} \bar{R}^k_{i_5i_6j} + i \leftrightarrow j) \right. \\ &\quad + \frac{11}{2} D_{i_1} \bar{R}_{ii_2i_3k} D_{i_4} \bar{R}^k_{i_5i_6j} \\ &\quad \left. + \frac{8}{5} \bar{R}_{ii_1i_2k} \bar{R}^k_{i_3i_4l} \bar{R}^l_{i_5i_6j} \right] q^{i_1} \cdots q^{i_6} (\dot{q}^i \dot{q}^j + b^i c^j + a^i a^j) \rangle (\sigma) d\sigma > \end{aligned} \quad (33)$$



$$\begin{aligned} &= \frac{5}{7!} (\beta\hbar)^3 \left\{ \frac{1}{120} \left(\square^2 \bar{R} + 2D_a \square D^a \bar{R} \right) + \frac{1}{60} \left(\square D_a D_b \bar{R}^{ab} + D_a D_b \square \bar{R}^{ab} \right. \right. \\ &\quad \left. \left. + D_a D_b D^a D_c \bar{R}^{bc} + D_a \square D_b \bar{R}^{ab} + D_a D_b D_c D^b \bar{R}^{ac} + D_a D_b D_c D^a \bar{R}^{bc} \right) \right. \\ &\quad \left. - \frac{1}{120} \frac{34}{5} \left[\bar{R}^{ab} \square \bar{R}_{ab} + \frac{3}{2} \bar{R}^{abcd} \square \bar{R}_{abcd} + 2\bar{R}^{ab} D^c D^d \bar{R}_{dacb} + 2\bar{R}^{abcd} D_d D_b \bar{R}_{ac} \right. \right. \\ &\quad \left. \left. + 12\bar{R}^{abcd} D_{(e} D_{d)} \bar{R}^e_{abc} \right] - \frac{1}{120} \frac{11}{2} \left[\left(D_a \bar{R}_{bc} \right)^2 + 4D_a \bar{R}_{bc} D^d \bar{R}_{dcab} + \frac{3}{2} \left(D_a \bar{R}_{bcde} \right)^2 \right. \right. \\ &\quad \left. \left. + 3D_a \bar{R}^{abcd} D^e \bar{R}_{ebcd} + 3D^a \bar{R}^{bcde} D_e \bar{R}_{bcd a} \right] + \frac{1}{120} \frac{8}{5} \left[\bar{R}_{ab} \bar{R}_c^b \bar{R}^{ac} + 9\bar{R}^{ab} \bar{R}_a^{edc} \bar{R}_{bcde} \right. \right. \\ &\quad \left. \left. - \bar{R}_{abcd} \bar{R}_e^a \bar{R}_f^c \bar{R}_{bedf} + \frac{7}{2} \bar{R}_{abcd} \bar{R}_{ef}^{ab} \bar{R}^{efcd} \right] \right\} \end{aligned}$$

⁵The recipe the diagrams have been drawn is: A line closing on a single vertex is represented by a Green function $G(\sigma, \sigma)$ with one integration variable while a line connecting two vertices stands for $G(\sigma, \tau)$ and indicates two integration variables.

2.

$$\begin{aligned}
I_2 &= -\beta\hbar \frac{1}{8} < \int_{-1}^0 \left[\left[\frac{1}{4!} (1-4\xi) D_{i_1} \cdots D_{i_4} \bar{R} - \frac{1}{3} \bar{R}^i{}_{i_1 i_2}{}^j \partial_{i_3} \bar{\Gamma}^l{}_{ik} \partial_{i_4} \bar{\Gamma}^k{}_{lj} \right. \right. \\
&+ \frac{1}{4} \bar{g}^{ij} \partial_{i_1} \partial_{i_2} \Gamma^l{}_{ik} \partial_{i_3} \partial_{i_4} \Gamma^k{}_{lj} \\
&+ \left. \left. \frac{\bar{g}^{ij}}{3!} \left(\partial_{i_1} \cdots \partial_{i_3} \Gamma^l{}_{ik} \partial_{i_4} \Gamma^k{}_{lj} + \partial_{i_1} \Gamma^l{}_{ik} \partial_{i_2} \cdots \partial_{i_4} \Gamma^k{}_{lj} \right) \right] q^{i_1} \cdots q^{i_4} \right] (\sigma) d\sigma > \quad (34)
\end{aligned}$$

$$\begin{aligned}
\bigcirc \bigcirc &= -(\beta\hbar)^3 \frac{1}{8} \left[\frac{1}{720} (1-4\xi) \left(3\Box^2 \bar{R} - 2D_a \bar{R}^{ab} D_b \bar{R} - 2\bar{R}^{ab} D_a D_b \bar{R} \right) \right. \\
&- \frac{1}{540} \left(\frac{1}{2} \bar{R}^{ab} \bar{R}_a{}^{cde} \bar{R}_{bcde} + \frac{1}{3} \bar{R}^{abcd} \bar{R}_a{}^e{}_c{}^f \bar{R}_{bedf} + \frac{13}{12} \bar{R}^{abcd} \bar{R}_{ab}{}^{ef} \bar{R}_{cdef} \right) \\
&- \frac{1}{1728} D^a \bar{R}^{bc} D_d \bar{R}_{ab}{}^d{}_c + \frac{1}{17280} \left(D^a \bar{R}^{bc} \right)^2 - \frac{5}{2304} D_a \bar{R}^{abcd} D_e \bar{R}^e{}_{bcd} \\
&+ \frac{1}{5760} D^a \bar{R}^{bc} D_c \bar{R}_{ab} - \frac{1}{480} \left(D^a \bar{R}^{bcde} \right)^2 - \frac{1}{360} D^a \bar{R}^{bcde} D_e \bar{R}_{bcda} \\
&+ \frac{1259}{518400} \bar{R}^{abcd} \bar{R}_{ab}{}^{ef} \bar{R}_{cdef} + \frac{47}{86400} \bar{R}^{ab} \bar{R}_a{}^{cde} \bar{R}_{bcde} - \frac{1}{7200} \bar{R}^{abcd} D_d D_b \bar{R}_{ac} \\
&- \frac{23}{4800} \bar{R}^{abcd} D_d D_e \bar{R}^e{}_{abc} - \frac{7}{3200} \bar{R}^{abcd} \Box \bar{R}_{abcd} - \frac{7}{1600} \bar{R}^{abcd} D_e D_d \bar{R}^e{}_{abc} \\
&\left. + \frac{209}{129600} \bar{R}^{abcd} \bar{R}_a{}^e{}_c{}^f \bar{R}_{bedf} \right]
\end{aligned}$$

3.

$$\begin{aligned}
I_3 &= \frac{1}{2} \left(\frac{1}{\beta\hbar} \right)^2 < \int_{-1}^0 \int_{-1}^0 \left[\left(\frac{1}{12} \right)^2 \left[D_{i_1} \bar{R}_{ii_2 i_3 j} q^{i_1} \cdots q^{i_3} \left(\dot{q}^i \dot{q}^j + b^i c^j + a^i a^j \right) \right] (\sigma) \right. \\
&\times \left[D_{i_4} \bar{R}_{ki_5 i_6 l} q^{i_4} \cdots q^{i_6} \left(\dot{q}^k \dot{q}^l + b^k c^l + a^k a^l \right) \right] (\tau) \\
&+ \frac{1}{5!} \left[\bar{R}_{ii_1 i_2 j} q^{i_1} q^{i_2} \left(\dot{q}^i \dot{q}^j + b^i c^j + a^i a^j \right) \right] (\sigma) \\
&\times \left[\left(D_{i_3} D_{i_4} \bar{R}_{ki_5 i_6 l} + \frac{8}{9} \bar{R}_{ki_3 i_4 m} \bar{R}^m{}_{i_5 i_6 l} \right) q^{i_3} \cdots q^{i_6} \left(\dot{q}^k \dot{q}^l + b^k c^l + a^k a^l \right) \right] (\tau) \Big] \\
&d\sigma d\tau > \quad (35)
\end{aligned}$$

$$\begin{aligned}
&\bigcirc + \bigcirc + \bigcirc + \left[\bigcirc \times \bigcirc \right] + \bigcirc + \bigcirc \\
&= -\frac{1}{2} \frac{1}{144} (\beta\hbar)^3 \left[\frac{1}{10} D_a \bar{R}_{bcde} D^e \bar{R}^{bcda} + \frac{1}{10} D_a \bar{R}^{abcd} D^e \bar{R}_{ebcd} + \frac{1}{20} \left(D_a \bar{R}_{bcde} \right)^2 \right. \\
&- \frac{2}{15} D_a \bar{R}_{bc} D_d \bar{R}^{abcd} + \frac{1}{30} \left(D_a \bar{R}_{bc} \right)^2 - \frac{1}{48} D_a \bar{R}^{ab} D_b \bar{R} - \frac{1}{48} D_a \bar{R}^{ab} D^c \bar{R}_{cb} - \frac{1}{192} \left(D_a \bar{R} \right)^2 \\
&+ \frac{1}{5!} \left(\frac{108}{15} \bar{R}_{abcd} \Box \bar{R}^{abcd} + \frac{72}{15} \bar{R}_{ab} \Box \bar{R}^{ab} - \frac{3}{2} \bar{R} \Box \bar{R} + \frac{432}{30} D_a D^b \bar{R}^{acde} \bar{R}_{bcde} + \frac{432}{30} D^a D_b \bar{R}^{bcde} \bar{R}_{acde} \right. \\
&+ \frac{144}{15} D_a D_b \bar{R}^{bcad} \bar{R}_{cd} + \frac{144}{15} D_a D_b \bar{R}_{cd} \bar{R}^{adbc} - 3 \bar{R} D_a D_b \bar{R}^{ab} + \frac{4}{3} \bar{R} \left(\bar{R}_{ab} \right)^2 - \frac{192}{5} \bar{R}_{abcd} \bar{R}^{bcde} \bar{R}_e^a \\
&\left. + \frac{64}{15} \bar{R}_{abcd} \bar{R}^b{}^d{}_e{}^f \bar{R}^{eafc} - \frac{224}{15} \bar{R}_{abcd} \bar{R}^{abef} \bar{R}_{ef}^{cd} - \frac{64}{15} \bar{R}_{ab} \bar{R}_c^b \bar{R}^{ca} + 2 \bar{R} \left(\bar{R}_{abcd} \right)^2 \right]
\end{aligned}$$

4.

$$\begin{aligned}
I_4 &= -\frac{1}{8} < \int_{-1}^0 \int_{-1}^0 \left[\frac{1}{12} (1 - 4\xi) \left[\bar{R}_{ii_1 i_2 j} q^{i_1} q^{i_2} (\dot{q}^i \dot{q}^j + b^i c^j + a^i a^j) \right] (\sigma) \right. \\
&\quad \times \left[D_{i_3} D_{i_4} \bar{R} q^{i_3} q^{i_4} \right] (\tau) \\
&\quad + \frac{1}{6} \left[\bar{R}_{ii_1 i_2 j} q^{i_1} q^{i_2} (\dot{q}^i \dot{q}^j + b^i c^j + a^i a^j) \right] (\sigma) \\
&\quad \times \left[\bar{g}^{mn} \partial_{i_3} \bar{\Gamma}_{mk}^l \partial_{i_4} \bar{\Gamma}_{ln}^k q^{i_3} q^{i_4} \right] (\tau) \\
&\quad + \frac{1}{12} (1 - 4\xi) \left[D_{i_1} \bar{R}_{i_2 i_3 j} q^{i_1} \dots q^{i_3} (\dot{q}^i \dot{q}^j + b^i c^j + a^i a^j) \right] (\sigma) \\
&\quad \times \left[D_{i_4} \bar{R} q^{i_4} \right] (\tau) + \frac{3}{5!} (1 - 4\xi) \left[\left(D_{i_1} D_{i_2} \bar{R}_{mi_3 i_4 n} + \frac{8}{9} \bar{R}_{mi_1 i_2 k} \bar{R}_{i_3 i_4 n}^k \right) \right. \\
&\quad \times \left. q^{i_1} \dots q^{i_4} (\dot{q}^m \dot{q}^n + b^m c^n + a^m a^n) \right] (\sigma) \bar{R} d\sigma d\tau >
\end{aligned} \tag{36}$$

$$\begin{aligned}
&\left[\text{Diagram 1} \times \text{Diagram 2} \right] + \text{Diagram 3} + \text{Diagram 4} + \left[\text{Diagram 5} \times \bullet \right] \\
&= -(\beta\hbar)^3 \frac{1}{8} \left\{ \frac{7}{180} \frac{1}{24} \bar{R} \square \bar{R} - \frac{1}{6} \frac{1}{144} \bar{R} (\bar{R}_{abcd})^2 + \frac{7}{180} \left[\frac{1}{24} D_a \bar{R}^{ab} D_b \bar{R} + \frac{1}{48} (D_a \bar{R})^2 \right] \right. \\
&\quad \left. + \frac{7}{600} \left[\frac{1}{24} \bar{R} \square \bar{R} + \frac{1}{12} \bar{R} D_a D_b \bar{R}^{ab} - \frac{1}{27} \bar{R} (\bar{R}_{ab})^2 - \frac{1}{18} \bar{R} (\bar{R}_{abcd})^2 \right] \right\}
\end{aligned}$$

5.

$$\begin{aligned}
I_5 &= -\frac{1}{2} \left(\frac{\beta\hbar}{8} \right)^2 < \int_{-1}^0 \int_{-1}^0 (1 - 4\xi)^2 \left[\left[D_{i_1} \bar{R} q^{i_1} \right] (\sigma) \left[D_{i_2} \bar{R} q^{i_2} \right] (\tau) \right. \\
&\quad + \left[q^{i_1} q^{i_2} D_{i_1} D_{i_2} \bar{R} \right] (\sigma) \bar{R} \Big] \\
&\quad + 2(1 - 4\xi) \bar{R} \left[\bar{g}^{ij} \partial_{i_1} \bar{\Gamma}_{ik}^l \partial_{i_2} \bar{\Gamma}_{lj}^k q^{i_1} q^{i_2} \right] (\tau) d\sigma d\tau >
\end{aligned} \tag{37}$$

$$\text{Diagram 1} + \left[\text{Diagram 2} \times \bullet \right] = (\beta\hbar)^3 \frac{1}{128} \left[\left(\frac{7}{15} \right)^2 \left(\frac{1}{12} D_a \bar{R} D^a \bar{R} + \frac{1}{6} \bar{R} \square \bar{R} \right) - \frac{14}{15} \frac{1}{36} \bar{R} (\bar{R}_{abcd})^2 \right]$$

6.

$$\begin{aligned}
I_6 &= \frac{1}{1296} \left(\frac{1}{\beta\hbar} \right)^3 < \int_{-1}^0 \int_{-1}^0 \int_{-1}^0 \left[\bar{R}_{ii_1 i_2 j} q^{i_1} q^{i_2} (\dot{q}^i \dot{q}^j + b^i c^j + a^i a^j) \right] (\sigma) \\
&\quad \times \left[\bar{R}_{ki_3 i_4 l} q^{i_3} q^{i_4} (\dot{q}^k \dot{q}^l + b^k c^l + a^k a^l) \right] (\tau) \\
&\quad \times \left[\bar{R}_{mi_5 i_6 n} q^{i_5} q^{i_6} (\dot{q}^m \dot{q}^n + b^m c^n + a^m a^n) \right] (\rho) d\sigma d\tau d\rho >
\end{aligned} \tag{38}$$

$$\begin{aligned}
& \left[\text{Diagram 1} \right]^3 + \text{Diagram 2} + \left[\text{Diagram 3} \times \text{Diagram 4} \right] + \text{Diagram 5} \\
& + \left[\text{Diagram 6} \times \text{Diagram 7} \right] + \left(\text{Diagram 8} \equiv \text{Diagram 9} \right) + \text{Diagram 10} \\
& = (\beta\hbar)^3 \frac{1}{1296} \left[\frac{1}{64} \bar{R}^3 + \frac{37}{360} \bar{R}^{ab} \bar{R}^{cd} \bar{R}_{acbd} - \frac{3}{16} \left(\bar{R}_{abcd} \right)^2 \bar{R} - \frac{40751}{10080} \bar{R}^{ab} \bar{R}^{cde} \bar{R}_{bcde} \right. \\
& \left. - \frac{1}{8} \bar{R} \left(\bar{R}_{ab} \right)^2 - \frac{20911}{10080} \bar{R}^{abcd} \bar{R}^{ef} \bar{R}_{ab} \bar{R}_{cdef} + \frac{409}{280} \bar{R}^{abcd} \bar{R}^e{}_a{}^f{}_c \bar{R}_{bedf} - \frac{137}{2016} \bar{R}^{ab} \bar{R}_b^c \bar{R}_{ca} \right]
\end{aligned}$$

7.

$$\begin{aligned}
I_7 &= -\frac{1}{576} \frac{1}{\beta\hbar} (1 - 4\xi) < \int_{-1}^0 \int_{-1}^0 \int_{-1}^0 \left[\bar{R}_{ii_1 i_2 j} q^{i_1} q^{i_2} \left(\dot{q}^i \dot{q}^j + b^i c^j + a^i a^j \right) \right] (\sigma) \\
&\times \left[\bar{R}_{ki_3 i_4 l} q^{i_3} q^{i_4} \left(\dot{q}^k \dot{q}^l + b^k c^l + a^k a^l \right) \right] (\tau) \bar{R} d\sigma d\tau d\rho >
\end{aligned} \tag{39}$$

$$\begin{aligned}
& \left[\left(\text{Diagram 1} \right)^2 \times \bullet \right] + \left[\text{Diagram 2} \times \bullet \right] + \left[\text{Diagram 3} \times \bullet \right] \\
& = -(\beta\hbar)^3 \frac{1}{576} \frac{7}{15} \bar{R} \left[\frac{1}{16} \bar{R}^2 - \frac{1}{6} (R_{ab})^2 - \frac{1}{4} (R_{abcd})^2 \right]
\end{aligned}$$

8.

$$\begin{aligned}
I_8 &= \frac{1}{768} \beta\hbar (1 - 4\xi)^2 < \int_{-1}^0 \int_{-1}^0 \int_{-1}^0 \left[\bar{R}_{ii_1 i_2 j} q^{i_1} q^{i_2} \left(\dot{q}^i \dot{q}^j + b^i c^j + a^i a^j \right) \right] (\sigma) \\
&\times \bar{R} \bar{R} d\sigma d\tau d\rho >
\end{aligned} \tag{40}$$

$$\left[\bullet \right]^2 \times \text{Diagram 1} = (\beta\hbar)^3 \frac{1}{768} \left(\frac{7}{15} \right)^2 \frac{1}{4} \bar{R}^3$$

9.

$$I_9 = \frac{1}{3!} \left(\frac{-\beta\hbar}{8} \right)^3 (1 - 4\xi)^3 < \int_{-1}^0 \int_{-1}^0 \int_{-1}^0 \bar{R} \bar{R} \bar{R} d\sigma d\tau d\rho > \tag{41}$$

$$\left[\bullet \right]^3 = -\frac{1}{3072} \left(\frac{7}{15} \right)^3 (\beta\hbar)^3 \bar{R}^3$$

Some comments are in order

- There are Feynman diagrams that satisfy the factorization property according to which a diagram breaks down into simpler subdiagrams. The four dimensional building block diagrams are depicted with their corresponding contributions in what follows:

$$\begin{aligned}
\text{Diagram 1: Two circles joined at a point} &= \frac{1}{4} \bar{R}^2 \\
\text{Diagram 2: Three circles joined at two points} &= -\frac{1}{6} (R_{ab})^2 \\
\text{Diagram 3: Two circles joined at two points (figure-eight)} &= -\frac{1}{4} (R_{abcd})^2 \\
\text{Diagram 4: Three circles joined at a central point} &= \alpha \left(\frac{1}{16} \bar{R} (\bar{R}_{abcd})^2 + \frac{1}{24} \bar{R} (\bar{R}_{ab})^2 \right) + \beta \frac{1}{12} \square \bar{R}
\end{aligned}$$

- Let us examine now the contribution of $\bar{R}^{\frac{n}{2}}$ terms. In n=2 dimensions the trace anomaly is:

$$\alpha_2 = -\frac{1}{24\pi} \bar{R}. \quad (42)$$

The situation changes in four dimensions in which the \bar{R}^2 terms, stemming from the interactions:

$$\begin{aligned}
I_{\bar{R}^2} &= -\frac{1}{144} \bar{R}_{ii_1 i_2 j} \bar{R} < \int_{-1}^0 q^{i_1} q^{i_2} (\dot{q}^i \dot{q}^j + b^i c^j + a^i a^j) d\sigma > \\
&+ \frac{1}{72 (\beta \hbar)^2} \bar{R}_{ki_1 i_2 l} \bar{R}_{mi_3 i_4 n} < \int_{-1}^0 \int_{-1}^0 \left[q^{i_1} q^{i_2} (\dot{q}^k \dot{q}^l + b^k c^l + a^k a^l) \right] (\sigma) \\
&\times \left[q^{i_3} q^{i_4} (\dot{q}^m \dot{q}^n + b^m c^n + a^m a^n) \right] (\tau) d\sigma d\tau > \\
&+ \frac{(\beta \hbar)^2}{1152} \bar{R}^2
\end{aligned} \quad (43)$$

and represented by the following diagrams:

$$\begin{aligned}
\text{Diagram 1: Two circles joined at a point} &= -(-\beta \hbar)^2 \left(\frac{1}{24} \right)^2 \bar{R}^2 \\
\text{Diagram 2: Three circles joined at two points} &= (-\beta \hbar)^2 \left[\frac{1}{2} \left(\frac{1}{24} \right)^2 \bar{R}^2 - \frac{1}{432} (\bar{R}_{ab})^2 \right] \\
\text{Diagram 3: A dot} &= (-\beta \hbar)^2 \frac{1}{2} \left(\frac{1}{24} \right)^2 \bar{R}^2
\end{aligned}$$

produce a vanishing result.

The same observation holds for the \bar{R} terms created by the interaction:

$$\begin{aligned}
I_{\bar{R}} &= \frac{1}{\beta \hbar} \frac{1}{6} \bar{R}_{ii_1 i_2 j} < \int_{-1}^0 q^{i_1} q^{i_2} (\dot{q}^i \dot{q}^j + b^i c^j + a^i a^j) d\sigma > \\
&- \beta \hbar \frac{1}{24} \bar{R}
\end{aligned} \quad (44)$$

and associated with the Feynman diagrams

$$\bigcirc \bigcirc + \bullet = \beta \hbar \bar{R} \left[\frac{1}{24} - \frac{1}{24} \right]$$

One making use of the identities (9)-(24) of the appendix and the 17 linearly independent terms listed below:

$$\begin{aligned} & R^3, \quad R(R_{ab})^2, \quad R(R_{abcd})^2, \quad R_{ab}R_c^bR^{ac}, \quad R^{ab}R^{cd}R_{acbd} \\ & R_{ab}R^{acde}R_{cde}^b, \quad R_{abcd}R^{abef}R_{ef}^{cd}, \quad R^{abcd}R_a^e{}^f{}_cR_{bedf}, \quad R\Box R \\ & R_{ab}\Box R^{ab}, \quad R_{abcd}\Box R^{abcd}, \quad (D_a R_{bc})^2, \quad R^{ab}D_b D_c R_a^c \\ & D^a R^{bc}D_b R_{ac}, \quad (D^a R^{bcde})^2, \quad \Box^2 R, \quad (D_a R)^2 \end{aligned} \quad (45)$$

can deduce for the integrated trace anomaly:

$$An_W(s=0, n) = \lim_{\beta \rightarrow 0} \left(\frac{1}{(2\pi\beta\hbar)^3} \int dx_0^i \prod_{i=1}^n \sqrt{g(x_0^i)} f(x_0^i) I(x_0^i) \right) \quad (46)$$

where

$$\begin{aligned} \frac{I(x_0^i)}{(\beta\hbar)^3} &= -\frac{1}{1296000} \bar{R}^3 + \frac{7}{129600} \bar{R} (\bar{R}_{ab})^2 + \frac{1}{43200} \bar{R} (\bar{R}_{abcd})^2 - \frac{5293}{13063680} \bar{R}_{ab} \bar{R}_c^b \bar{R}^{ac} \\ &+ \frac{9287}{16329600} \bar{R}^{ab} \bar{R}^{cd} \bar{R}_{acbd} - \frac{159421}{130636800} \bar{R}_{ab} \bar{R}^{acde} \bar{R}_{cde}^b - \frac{18413}{26127360} \bar{R}_{abcd} \bar{R}^{abef} \bar{R}_{ef}^{cd} \\ &+ \frac{661}{362880} \bar{R}^{abcd} \bar{R}_a^e{}^f{}_c \bar{R}_{bedf} + \frac{1}{21600} \bar{R} \Box \bar{R} - \frac{3}{5600} \bar{R}_{ab} \Box \bar{R}^{ab} \\ &- \frac{191}{483840} \bar{R}_{abcd} \Box \bar{R}^{abcd} + \frac{7}{8640} D^a \bar{R}^{bc} D_b \bar{R}_{ac} - \frac{1}{20160} (D^a \bar{R}^{bcde})^2 \\ &- \frac{17}{100800} \Box^2 \bar{R} - \frac{1}{5040} \bar{R}^{ab} D_b D_c \bar{R}_a^c - \frac{127}{120960} (D_a \bar{R}_{bc})^2 - \frac{67}{604880} (D_a \bar{R})^2. \end{aligned} \quad (47)$$

5 The computer algebra program

Vertices I_1 to I_9 were calculated using the Riegeom package [18], which is a Maple package for manipulating generic symbolic tensor expressions in the context of Riemannian geometry. In this section we show the process of calculating the vertex I_7 . The lines beginning with “>” in typewrite font are the input in a Maple worksheet. A colon at the end of the command hides the result. We start loading the package

> with(Riegeom);

defining Christoffel(Gamma), Riemann(R), Weyl(C), Ricci(R), LeviCivita(eta),

TraceFreeRicci(S) for Dimension = 4, CoordinateName = X, MetricName = g

[absorb \bar{g} , changedumind, cleartensor, codiff, coordinate, definetensor, dimension,
expandcodiff, lptensor, ltensor, metric, normalform, off, on, printtensor, replace,
simpLC, simptensor, sreplace, switches, symmetrize, symmetry, tdiff]

Next command reads file “Vertex.mpl” that contains procedures written specifically for the calculation of vertices I_1 to I_9 using Maple programming language and Riegeom commands.

```
> read('Vertex.mpl');
```

We setup spacetime dimension using Riegeom command `dimension`.

```
> dimension(6):
```

Next command enters the tensor coefficient of vertex I_7 .

```
> tensor_coeff := printtensor(R[-mu1,-i1,-i2,-nu1]*R[-mu2,-i3,-i4,-nu2]);
```

$$tensor_coeff := R_{\mu_1 i_1 i_2 \nu_1} R_{\mu_2 i_3 i_4 \nu_2}$$

`Printtensor` is the Riegeom interface command. Indices with minus sign are covariant and with plus sign are contravariant. In the next command we enter the field terms. We use a Maple list (which preserves order) instead of an expression written in terms of the commuting product operator. In this form we have full control over the order of the fields.

```
> L := printtensor([[q[i1],q[i2]](sigma),[qdot[mu1],qdot[nu1]],
> [b[mu1],c[nu1]], [a[mu1],a[nu1]]](sigma),[q[i3],q[i4]](tau),
> [[qdot[mu2],qdot[nu2]], [b[mu2],c[nu2]], [a[mu2],a[nu2]]](tau)]);
```

$$L := [[q^{i1}(\sigma), q^{i2}(\sigma)], \\ [[qdot^{\mu1}(\sigma), qdot^{\nu1}(\sigma)], [b^{\mu1}(\sigma), c^{\nu1}(\sigma)], [a^{\mu1}(\sigma), a^{\nu1}(\sigma)]], \\ [q^{i3}(\tau), q^{i4}(\tau)], \\ [[qdot^{\mu2}(\tau), qdot^{\nu2}(\tau)], [b^{\mu2}(\tau), c^{\nu2}(\tau)], [a^{\mu2}(\tau), a^{\nu2}(\tau)]]]$$

Next command finds all independent ⁶ index configurations that contribute to the final result.

```
> Lic := ind_config([i1,i2,mu1,nu1,i3,i4,mu2,nu2]):
```

```
> N := nops(Lic);
```

$$N := 105$$

In the case of 8 field indices, there are 105 independent index configurations. We show the first 3 ones.

```
> for i to 3 do evaln(Lic[i])=Lic[i] od;
```

$$Lic_1 = [i1, i2, \mu1, \nu1, i3, i4, \mu2, \nu2]$$

$$Lic_2 = [i1, i2, \mu1, \nu1, i3, \mu2, i4, \nu2]$$

$$Lic_3 = [i1, i2, \mu1, \nu1, i3, \nu2, \mu2, i4]$$

Next command is a loop over the 105 index configurations. The results are stored in a table of results called `res`.

```
> for i to N do
```

```
>   expr := WickContractions(L, Lic[i]):
```

```
>   res[i] := Vertex(tensor_coeff*expr):
```

```
> od:
```

Command `WickContractions` performs the Wick contractions given by eq. (27) to (32). We

⁶The number of independent index configurations is given by $\frac{n!}{(\frac{n}{2})! 2^{\frac{n}{2}}}$ where $(\frac{n}{2})!$ represents the possible permutations of index pairs and $2^{\frac{n}{2}}$ the permutation of indices within the pairs.

show an example with the fourth index configuration.

```
> expr := factor(WickContractions(L, Lic[4]));
```

$$expr := \beta^4 hbar^4 g^{i1 i2} g^{\mu2 \nu2} g^{\mu1 i3} g^{\nu1 i4} \sigma(1 + \sigma)(\tau + \text{Heaviside}(\sigma - \tau))^2$$

Command **Vertex** simply multiplies the result of Wick contractions to the tensor coefficient, simplifies the tensor expression, and isolates the terms to be integrated, since we noticed that, for the most complicate vertices, Maple spends more time integrating Dirac delta and step functions than simplifying the tensorial terms.

```
> Vertex(tensor_coeff*expr);
```

$$[\beta^4 hbar^4 R^{\nu2 \nu1} R_{\nu2 \nu1}, \sigma(1 + \sigma)(\tau + \text{Heaviside}(\sigma - \tau))^2]$$

Next command adds the results of all index configurations after integrating over variables σ and τ . Functions **int_tau** and **int_sigma** replace the usual Maple integrator, since Maple **int** command fails to return the correct value for complicated vertices.

```
> final_value := simptensor(add(res[i][1]*
```

```
> int_tau(int_sigma(res[i][2])), i=1..N));
```

$$final_value := \frac{1}{16} R^2 \beta^4 hbar^4 - \frac{1}{6} \beta^4 hbar^4 R^{\nu1 \mu2} R_{\nu1 \mu2} - \frac{1}{4} \beta^4 hbar^4 R^{\nu2 i1 -b0 -c0} R_{\nu2 i1 -b0 -c0}$$

Collecting $\hbar^4 \beta^4$, we obtain the final form.

```
> collect(final_value, [beta, hbar]);
```

$$\left(\frac{1}{16} R^2 - \frac{1}{6} R^{\nu1 \mu2} R_{\nu1 \mu2} - \frac{1}{4} R^{\nu2 i1 -b0 -c0} R_{\nu2 i1 -b0 -c0}\right) hbar^4 \beta^4$$

For vertices I_2 and I_6 , the method described above has some extra complications. Vertex I_6 has a large number of index configurations ($N := 10395$) and vertex I_2 uses huge side identities obtained in Riemann normal coordinates as described in appendix A.3.

This computer algorithm can easily be extended to higher dimensions to predict the trace anomalies in 8 and 10 dimensions. Of course there are strong limitations implied by the exponential growth of the problem which can be relaxed by increasing the available computing power. The tensorial upper bound of the program is reached by the product of eight Riemann curvature tensors but such a case is beyond the scope of the present work.

6 Conclusions

In this article we calculate the integrated trace anomaly for a real spin-0 scalar field living on a curved six dimensional manifold. This is achieved by relying on a recursive computation of the metric tensor components in Riemann normal coordinates. One can use the general formulae (9) and (11) to reach the desired order of metric expansion induced by the dimensionality of the manifold. Adopting the path integral formalism of quantum mechanical non-linear sigma models, we evaluate all the vertices and the corresponding Feynman diagrams that contribute to the present anomaly. A computer based program is used to perform the otherwise inevitable task of integration over distributions and contractions of the various tensors involved. The final result derived in this process involves 17 scalar monomials consisting of covariant derivatives and/or Riemann tensors.

Appendix

A.1 Useful identities

We consider a Riemannian (or pseudo-Riemannian) manifold equipped with its standard (metric compatible, torsion free) connection. The Riemann curvature is defined as:

$$R^a{}_{bcd} = \partial_c \Gamma^a_{bd} + \Gamma^e_{bd} \Gamma^a_{ec} - c \leftrightarrow d \quad (1)$$

possessing the familiar symmetries:

- antisymmetry

$$R_{abcd} = -R_{bacd} = -R_{abdc} \quad (2)$$

- pair symmetry

$$R_{abcd} = R_{badc} \quad (3)$$

- cyclic symmetry

$$R_{abcd} + R_{adbc} + R_{acdb} = 0 \quad (4)$$

- Bianchi symmetry and the related identities

$$R_{abcd;e} + R_{abec;d} + R_{abde;c} = 0 \quad (5)$$

$$R_{bc;a} - R_{ab;c} + R^d{}_{bca;d} = 0 \quad (6)$$

$$\left(R^a_b - \frac{1}{2} \delta^a_b R \right)_{;a} = 0 \quad (7)$$

The Ricci curvature and scalar are:

$$R_{ab} = R^q{}_{aqb}; \quad R = R^q{}_q \quad (8)$$

In this paper the following identities have been exploited to simplify our expressions:

$$R_{abcd} R^{acbd} = \frac{1}{2} R_{abcd}^2 \quad (9)$$

$$R_{abcd} R^{ecbd} = \frac{1}{2} R_{abcd} R^{ebcd} \quad (10)$$

$$R_{a(bc)d} R^{e(af)b} R_{f\ e}{}^{(d\ c} = \frac{1}{8} R_{abcd} \left[R^e{}_{af}{}^c R^b{}_{ed}{}^f + \frac{7}{2} R^{abef} R_{ef}{}^{cd} \right] \quad (11)$$

$$\square D_a D_b R^{ab} = \frac{1}{2} \square^2 R \quad (12)$$

$$\begin{aligned} D_a D_b \square R^{ab} &= \frac{1}{2} \square^2 R - \frac{1}{2} (D_a R)^2 - 2 R_{ab} D^a D^b R \\ &+ 2 R_{ab} R^b{}_c R^{ac} - 2 R^{ab} R^{cd} R_{acbd} - 4 D_a R_{bc} D^b R^{ac} \\ &+ 3 (D_a R_{bc})^2 + \frac{1}{2} R_{abcd} \square R^{abcd} - 2 R_{abcd} R^e{}_{af}{}^c R^{be}{}_{df} \end{aligned}$$

$$+ 2R^{ab}R^{cde}{}_a R_{bcde} - \frac{1}{2}R^{abcd}R_{abef}R_{cd}{}^{ef} + R_{ab}\square R^{ab} \quad (13)$$

$$D_a D_b D^a D_c R^{bc} = \frac{1}{2}\square^2 R - \frac{1}{4}(D_a R)^2 - \frac{1}{2}R_{ab}D^a D^b R \quad (14)$$

$$\begin{aligned} D_a D_b D_c D^b R^{ac} &= \frac{1}{2}\square^2 R - \frac{1}{2}(D_a R)^2 - 2R_{ab}D^a D^b R \\ &+ R^{abcd}D_d D_b R_{ac} + 2R_{ab}R_c^b R^{ac} - 2R^{ab}R^{cd}R_{acbd} \\ &- 3D_a R_{bc}D^b R^{ac} + R_{ab}\square R^{ab} + 2(D_a R_{bc})^2 \end{aligned} \quad (15)$$

$$\begin{aligned} D_a D_b D_c D^a R^{bc} &= \frac{1}{2}\square^2 R - \frac{1}{2}(D_a R)^2 - 2R_{ab}D^a D^b R \\ &+ 2R_{ab}R_c^b R^{ac} - 2R^{ab}R^{cd}R_{acbd} - 3D_a R_{bc}D^b R^{ac} \\ &+ R_{ab}\square R^{ab} + 2(D_a R_{bc})^2 + \frac{1}{4}R_{abcd}\square R^{abcd} \\ &- R^{abcd}R^e{}_{af}{}^c R_{df}{}^{be} + R^{ab}R^{cde}{}_a R_{bcde} - \frac{1}{4}R^{abcd}R_{abef}R_{cd}{}^{ef} \end{aligned}$$

$$D^a R^{bc}D_d R_{ab}{}^d{}_c = (D_a R_{bc})^2 - D_a R_{bc}D^b R^{ac} \quad (16)$$

$$R^{abcd}D_d D_b R_{ac} = \frac{1}{4}R^{abcd}\square R_{abcd} + \frac{1}{2}R_{ab}R^{acde}R^b{}_{cde} - R_{abcd}R^e{}_{af}{}^c R^{bf}{}_{de} \quad (17)$$

$$R^{ab}D^c D_d R_{ca}{}^d{}_b = R^{ab}\square R_{ab} - \frac{1}{2}R^{ab}D_a D_b R + R^{ab}R_a^c R_{cb} - R^{ab}R^{cd}R_{acbd} \quad (18)$$

$$R^{abcd}D_d D_e R^e{}_{abc} = R^{abcd}D_d D_b R_{ac} \quad (19)$$

$$D_a R^{abcd}D_e R^e{}_{bcd} = 2[(D_a R_{bc})^2 - D_a R_{bc}D^b R^{ac}] \quad (20)$$

$$R^{abcd}D_e D_d R^e{}_{abc} = \frac{1}{4}R^{abcd}\square R_{abcd} \quad (21)$$

$$D^a R^{bcd}D_e R_{abcd} = \frac{1}{2}(D_a R_{bcde})^2 \quad (22)$$

$$D_a R D^a R = \frac{1}{2}\square R^2 - R\square R \quad (23)$$

$$R^{ab}D_b D_a R = 2R^{ab}D^c D_b R_{ac} - 2R^{ab}R^{cd}R_{acbd} + 2R^{ab}R_a^c R_{bc} \quad (24)$$

Proof of (9) By cyclic symmetry,

$$R_{abcd}R^{acbd} = -R_{abcd}(R^{adcb} + R^{abdc}) = -R_{abcd}R^{adcb} + R_{abcd}^2. \quad (25)$$

Rename the indices in the first term:

$$R_{abcd}R^{adcb} = R_{abdc}R^{acdb} = R_{abcd}R^{acbd} \quad (26)$$

Solve for the desired object. In the same vein we can prove (10) as well as $D_a R^{abcd}D^e R_{ecbd} = \frac{1}{2}(D_a R^{abcd})^2$ and $R^{abcd}\square R_{abcd} = \frac{1}{2}R^{abcd}\square R_{abcd}$.

Proof of (11) Expanding out the products, the resulting sum can be expressed in terms of the invariants:

$$\begin{aligned} L_1 &= R_{abcd}R^{abef}R^c{}_{ef}{}^d \\ L_2 &= R_{abcd}R^{abef}R_{ef}{}^{cd} \\ L_3 &= R_{abcd}R^e{}_{fa}{}^c R^f{}_{de}{}^b \\ L_4 &= R_{abcd}R^e{}_{af}{}^c R^b{}_{ed}{}^f \end{aligned} \quad (27)$$

and with the assistance of the identities:

$$\begin{aligned}
L_1 &= \frac{1}{2}L_2 \\
L_3 &= \frac{1}{4}L_2 \\
L_4 &= -L_3 + R_{abcd}R_{af}^e R_{fd}^b{}^e
\end{aligned} \tag{28}$$

one can recover the proposed formula.

Proof of (12) - (24) With the assistance of (7) - (9) it is a straightforward exercise to show their validity.

A.2 Six dimensional invariants

In d-dimensions the polynomials $C^{abcd}C_{cdef}C_{ab}^{ef}$ and $C_{abcd}C^{ebcf}C_{ef}^a{}^d$ (type-B anomaly according to the geometric classification of [6]) are written as:

$$\begin{aligned}
Inv_1 = C^{abcd}C_{cdef}C_{ab}^{ef} &= R^{abcd}R_{cdef}R_{ab}^{ef} - \frac{12}{d-2}R_{abcd}R^{abce}R_e^d + \frac{6}{(d-1)(d-2)}RR_{abcd}^2 \\
&+ \frac{8(2d-3)}{(d-1)^2(d-2)^3}R^3 - \frac{24(2d-3)}{(d-1)(d-2)^3}RR_{ab}^2 \\
&+ \frac{16(d-1)}{(d-2)^3}R^{ab}R_{bc}R_a^c + \frac{24}{(d-2)^2}R^{abcd}R_{ac}R_{bd}.
\end{aligned} \tag{29}$$

$$\begin{aligned}
Inv_2 = C_{abcd}C^{ebcf}C_{ef}^a{}^d &= \frac{(d^2+d-4)}{(d-1)^2(d-2)^3}R^3 - \frac{3(d^2+d-4)}{(d-1)(d-2)^3}RR_{ab}^2 \\
&+ \frac{3}{2(d-1)(d-2)}RR_{abcd}^2 + \frac{2(3d-4)}{(d-2)^3}R_{ab}R^{bc}R_c^a \\
&+ \frac{3d}{(d-2)^2}R_{abcd}R^{ac}R^{bd} - \frac{3}{d-2}R_{abcd}R^{abce}R_e^d \\
&+ R_{abcd}R^{ebcf}R_{ef}^a{}^d.
\end{aligned} \tag{30}$$

Expressions(29) and(30) reproduce in $d = 6$ the unique already known Weyl invariant polynomials of [4] constructed only out of Weyl tensors. In higher dimensions the independent ways to contract indices among a number of Weyl tensors increases.

Another invariant made out of covariant derivatives of the Weyl tensor is:

$$\begin{aligned}
Inv_3 &= -R^3 + 8RR^{ab}R_{ab} + 2RR^{abcd}R_{abcd} - 10R^{ab}R_b^c R_{ca} - 10R_{ab}R^{acdb}R_{cd} \\
&+ \frac{1}{2}R\Box R - 5R^{ab}\Box R_{ab} + 5R^{abcd}\Box R_{abcd}.
\end{aligned} \tag{31}$$

The final nontrivial ⁷ invariant (type-A) we would like to consider stems from the Euler form which exists in any even dimension $d = 2n$:

$$E_{2n} = \frac{1}{(4\pi)^n n!} \int_{\mathcal{M}} \epsilon_{a_1 a_2 \dots a_{2n}} R^{a_1 a_2} \wedge \dots \wedge R^{a_{2n-1} a_{2n}}$$

⁷The terminology trivial invariants is adopted to justify the existence of covariant total derivatives of polynomials over Riemann tensor and it's covariant derivatives. These invariants coincide with variations of all independent local counterterms to effective action.

$$\begin{aligned}
&= \frac{1}{(4\pi)^n} \frac{1}{n!} \frac{1}{2^n} \int_{\mathcal{M}} \epsilon_{a_1 a_2 \dots a_{2n}} \epsilon^{b_1 b_2 \dots b_{2n}} R^{a_1 a_2}_{b_1 b_2} \dots R^{a_{2n-1} a_{2n}}_{b_{2n-1} b_{2n}} dV \\
&= \frac{1}{(4\pi)^n} \frac{1}{n!} \int_{\mathcal{M}} \mathcal{E}_{2n} dV
\end{aligned} \tag{32}$$

where

$$\begin{aligned}
\mathcal{E}_{2n} &= \frac{1}{2^n} \epsilon_{a_1 a_2 \dots a_{2n}} \epsilon^{b_1 b_2 \dots b_{2n}} R^{a_1 a_2}_{b_1 b_2} \dots R^{a_{2n-1} a_{2n}}_{b_{2n-1} b_{2n}} \\
&= \frac{1}{2^n} \det(\delta_{a_i}^{b_i}) R^{a_1 a_2}_{b_1 b_2} \dots R^{a_{2n-1} a_{2n}}_{b_{2n-1} b_{2n}}; \quad i = 1, \dots, 2n
\end{aligned} \tag{33}$$

is the Euler number which is a total divergence, vanishes in all lower integer dimensions and $dV = e^1 \wedge \dots \wedge e^{2n} = \sqrt{-g} d^{2n}x$ is the volume form. In six dimensions the Euler number becomes:

$$\begin{aligned}
Inv_4 = \mathcal{E}_6 &= R^3 - 12 R R^{ab} R_{ab} + 16 R^{ab} R_b^c R_{ac} + 24 R^{ab} R^{cd} R_{acbd} + 3 R R^{abcd} R_{abcd} \\
&+ 4 R^{abcd} R_{ab}{}^{ef} R_{cdef} - 24 R^{ab} R_a{}^{cde} R_{bcde} - 8 R^{abcd} R_a{}^e{}_c{}^f R_{bedf}.
\end{aligned} \tag{34}$$

A.3 The vertex I_2

The key idea for calculating the contribution of this vertex is to express each interaction term in terms of polynomials of the Riemann curvature and its covariant derivatives. This can be achieved by making use of the identities(23) and the symmetries implied by the structure of each term separately.

- With the help of the identity $\partial_a \bar{\Gamma}_{bc}^d = \partial_a \bar{\Gamma}_{(bc)}^d = -\frac{2}{3} \bar{R}_{(bc)a}^d$ one may rewrite the second term of I_2 as ⁸:

$$-\frac{1}{3} \bar{R}^i{}_{i_1 i_2}{}^j \partial_{i_3} \bar{\Gamma}_{ik}^l \partial_{i_4} \bar{\Gamma}_{lj}^k = -\frac{4}{27} \bar{R}^i{}_{i_1 i_2}{}^j \bar{R}_{(ik)i_3}^l \bar{R}_{(lj)i_4}^k. \tag{35}$$

- Expanding out the identity $\partial_{(a} \partial_b \bar{\Gamma}_{c)d}^e = -\frac{1}{2} D_{(a} \bar{R}_{b|d|c)}^e$ we get:

$$\begin{aligned}
\partial_{i_1} \partial_{i_2} \bar{\Gamma}_{ki}^l + \partial_{i_1} \partial_k \bar{\Gamma}_{i_2 i}^l + \partial_{i_2} \partial_k \bar{\Gamma}_{i_1 i}^l &= -\frac{1}{4} (D_{i_1} \bar{R}_{i_2 ik}^l + D_{i_2} \bar{R}_{i_1 ik}^l + D_k \bar{R}_{i_1 i_2}^l \\
&+ D_k \bar{R}_{i_2 i i_1}^l + D_{i_1} \bar{R}_{k i i_2}^l + D_{i_2} \bar{R}_{k i i_1}^l).
\end{aligned} \tag{36}$$

Notice that the third term of I_2 is symmetric under the interchange of the lower indices of the Christoffel symbols namely $\mu \leftrightarrow k$ and $l \leftrightarrow \nu$. So in(36) we interchange these indices and add the two expressions together. We then obtain:

$$\begin{aligned}
&2 \left(\partial_{i_1} \partial_{i_2} \bar{\Gamma}_{ki}^l + \partial_{i_1} \partial_{(k} \bar{\Gamma}_{i) i_2}^l + \partial_{i_2} \partial_{(k} \bar{\Gamma}_{i) i_1}^l \right) \\
&= -\frac{1}{4} (D_{i_1} \bar{R}_{k i i_2}^l + D_{i_2} \bar{R}_{k i i_1}^l + D_{i_1} \bar{R}_{i k i_2}^l + D_{i_2} \bar{R}_{i k i_1}^l \\
&+ D_k \bar{R}_{i_2 i i_1}^l + D_k \bar{R}_{i_1 i i_2}^l + D_i \bar{R}_{i_2 k i_1}^l + D_i \bar{R}_{i_1 k i_2}^l).
\end{aligned} \tag{37}$$

⁸Identical results arise if one makes use of the relation $\partial_{(a} \bar{\Gamma}_{b)c}^d = -\frac{1}{3} \bar{R}_{(bca)}^d$ which communicates with the one employed in the text by the relation $\partial_{(a} \bar{\Gamma}_{bc)}^d = 0$.

We would like now to express the second and third terms of the lefthand side of(37) in terms of $\partial_{i_1}\partial_{i_2}\bar{\Gamma}_{k\mu}^l$ and covariant derivatives of the Riemann curvature. Starting from:

$$\begin{aligned} D_{i_1}\bar{R}_{i_2k}^l &= \partial_{i_1}\partial_{i_2}\bar{\Gamma}_{ik}^l - \partial_{i_1}\partial_i\bar{\Gamma}_{ki_2}^l \Rightarrow \\ \partial_{i_1}\partial_{(i}\bar{\Gamma}_{k)i_2}^l &= \partial_{i_1}\partial_{i_2}\bar{\Gamma}_{(ik)}^l - D_{i_1}\bar{R}_{(i|i_2|k)}^l \end{aligned} \quad (38)$$

one has:

$$\partial_{i_1}\partial_{(k}\bar{\Gamma}_{i)i_2}^l + \partial_{i_2}\partial_{(k}\bar{\Gamma}_{i)i_1}^l = 2\partial_{i_1}\partial_{i_2}\bar{\Gamma}_{ki}^l - D_{i_1}R_{(i|i_2|k)}^l - D_{i_2}R_{(i|i_1|k)}^l. \quad (39)$$

Substituting(39) into(37) one gets:

$$\begin{aligned} 3\partial_{i_1}\partial_{i_2}\bar{\Gamma}_{ki}^l &= -\frac{1}{8}(D_{i_1}\bar{R}_{kii_2}^l + D_{i_2}\bar{R}_{kii_1}^l + D_{i_1}\bar{R}_{iki_2}^l + D_{i_2}\bar{R}_{iki_1}^l \\ &+ D_k\bar{R}_{i_2ii_1}^l + D_k\bar{R}_{i_1ii_2}^l + D_i\bar{R}_{i_2ki_1}^l + D_i\bar{R}_{i_1ki_2}^l) \\ &+ \frac{1}{2}\left(D_{i_1}R_{ii_2k}^l + D_{i_1}R_{ki_2i}^l + D_{i_2}R_{ii_1k}^l + D_{i_2}R_{ki_1i}^l\right). \end{aligned} \quad (40)$$

Finally the expression we are going to use for the contribution of the third term in the action I_2 is:

$$\begin{aligned} \partial_{i_1}\partial_{i_2}\bar{\Gamma}_{ki}^l &= -\frac{1}{24}(D_{i_1}\bar{R}_{kii_2}^l + D_{i_2}\bar{R}_{kii_1}^l + D_{i_1}\bar{R}_{iki_2}^l + D_{i_2}\bar{R}_{iki_1}^l \\ &+ D_k\bar{R}_{i_2ii_1}^l + D_k\bar{R}_{i_1ii_2}^l + D_i\bar{R}_{i_2ki_1}^l + D_i\bar{R}_{i_1ki_2}^l) \\ &+ \frac{1}{6}\left(D_{i_1}R_{ii_2k}^l + D_{i_1}R_{ki_2i}^l + D_{i_2}R_{ii_1k}^l + D_{i_2}R_{ki_1i}^l\right). \end{aligned} \quad (41)$$

and a similar relation holds for the term $\partial_{i_3}\partial_{i_4}\bar{\Gamma}_{li}^k$.

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